Numer. Math. 50, 503-518 (1987)

Numerische
Mathematik

Moment-Preserving Spline Approximation on Finite Intervals*

Marco Frontini[1], Walter Gautschi[2], and Gradimir V. Milovanović[3]

[1] Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32, I-20133 Milano, Italy
[2] Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, USA
[3] Faculty of Electronic Engineering, Department of Mathematics, P.O. Box 73, Y-18000 Niš, Yugoslavia

Summary. Continuing previous wotk, we discuss the problem of approximating a function f on the interval $[0, 1]$ by a spline function of degree m, with n (variable) knots, matching as many of the initial moments of f as possible. Additional constraints on the derivatives of the approximation at one endpoint of $[0, 1]$ may also be imposed. We show that, if the approximations exist, they can be represented in terms of generalized Gauss-Lobatto and Gauss-Radau quadrature rules relative to appropriate moment functionals or measures (depending on f). Pointwise convergence as $n \to \infty$, for fixed $m > 0$, is shown for functions f that are completely monotonic on $[0, 1]$, among others. Numerical examples conclude the paper.

Subject Classifications: AMS: Primary 41A15, 65D32; Secondary 33A65; CR: G1.2, G1.4.

1. Introduction

In previous papers [4, 6] two of us dealt with the problem of approximating a given function f on $[0, \infty]$ by a spline function of fixed degree (with variable knots) in such a way as to reproduce as many moments of f as possible. Having had in mind applications to physics, our functions $f = f(r)$ were considered functions of the radial distance $r = \|x\|$ of a vector $x \in \mathbb{R}^d$, and accordingly the moments were "spherical moments". We now wish to consider the analogous problem on an arbitrary finite interval. In this case, the interpretation of the independent variable as a radial distance is no longer meaningful, and our functions $f = f(t)$, therefore, are now simply functions of a real variable t on some given interval $[a, b]$. The case of a semi-infinite interval having been treated in our previous work, we restrict attention here to the case of a finite

* The work of the first author was supported by the *Ministero della Pubblica Istruzione* and by the *Consiglio Nazionale delle Ricerche*. The work of the second author was supported, in part, by the National Science Foundation under grant DCR-8320561

interval, which can be standardized to $[a, b]=[0, 1]$. The case of the whole real line, $[a, b]=\mathbb{R}$, is also of interest, as is the case of periodic splines. Both, however, appear to be less amenable to the type of analysis we are going to give, and will not be considered here.

2. Spline Approximation on [0, 1]

A spline function of degree $m \geqq 0$, with n (distinct) knots $\tau_1, \tau_2, \ldots, \tau_n$ in the interior of $[0, 1]$, can be written in terms of truncated powers in the form

$$s_{n,m}(t)=p_m(t)+\sum_{\nu=1}^{n} a_\nu(\tau_\nu-t)_+^m, \quad 0 \leqq t \leqq 1, \tag{2.1}$$

where a_ν are real numbers and p_m is a polynomial of degree $\leqq m$. (Our choice of truncated powers distinguishes the right endpoint of $[0, 1]$ in the sense that $s_{n,m}(t) \equiv p_m(t)$, $t \geqq 1$.) We consider two related problems:

Problem I. Determine $s_{n,m}$ in (2.1) such that

$$\int_0^1 t^j s_{n,m}(t)\,dt=\int_0^1 t^j f(t)\,dt, \quad j=0, 1, \ldots, 2n+m. \tag{2.2}$$

*Problem I**. Determine $s_{n,m}$ in (2.1) such that

$$s_{n,m}^{(k)}(1)=f^{(k)}(1), \quad k=0, 1, \ldots, m, \tag{2.3}$$

and such that (2.2) holds for $j=0, 1, \ldots, 2n-1$. Here we must assume that f has m derivatives at $t=1$, all being known.

Both problems will be solved in two ways: first in terms of moment functionals, then in terms of Gauss-Christoffel quadrature. The former approach requires only the existence and knowledge of the moments of f involved; the latter requires additional regularity of f, but lends itself better to stable implementations.

2.1. Solution of Problems I and I by Moment Functionals*

We first consider Problem I. Let

$$\mu_j=\frac{(m+j+1)!}{m!\,j!}\int_0^1 t^j f(t)\,dt, \quad j=0, 1, \ldots, 2n+m, \tag{2.4}$$

where the moments of f on the right are assumed to exist. (They do, of course, if f is integrable on $[0, 1]$.) We define a linear functional $\mathscr{L}$ on the set of polynomials of the form $t^{m+1} p(t)$, $p \in \mathbb{P}_{2n+m}$, by

$$\mathscr{L}(t^{m+1} \cdot t^j)=\mu_j, \quad j=0, 1, \ldots, 2n+m. \tag{2.5}$$

Then the inner product

$$(p,q)=\mathscr{L}(t^{m+1}(1-t)^{m+1}p\cdot q) \tag{2.6}$$

is well defined for any polynomials p, q for which $p\cdot q\in\mathbb{P}_{2n-1}$. In particular, we can define (if it exists) the monic polynomial $\pi_n(\cdot)=\pi_n(\cdot;\mathscr{L})$ of degree n orthogonal with respect to the inner product (2.6) to all polynomials of lower degree,

$$\begin{aligned} &\deg \pi_n=n, \quad \pi_n(t)=t^n+\dots,\\ &(\pi_n,q)=0, \quad \text{all } q\in\mathbb{P}_{n-1}. \end{aligned} \tag{2.7}$$

Theorem 2.1. *There exists a unique spline function on* $[0,1]$,

$$s_{n,m}(t)=p_m(t)+\sum_{\nu=1}^{n}a_\nu(\tau_\nu-t)_+^m, \qquad 0<\tau_\nu<1,\ \tau_\nu\neq\tau_\mu \text{ for } \nu\neq\mu, \tag{2.8}$$

satisfying the $2n+m+1$ *moment equations* (2.2) *of Problem I if and only if the orthogonal polynomial* $\pi_n(\cdot)=\pi_n(\cdot;\mathscr{L})$ *in* (2.7) *exists uniquely and has n distinct real zeros* $\tau_\nu^{(n)}$, $\nu=1,2,\dots,n$, *all contained in the open interval* (0, 1). *The knots* τ_ν *in* (2.8) *are then precisely these zeros,*

$$\tau_\nu=\tau_\nu^{(n)}, \qquad \nu=1,2,\dots,n, \tag{2.9}$$

while the coefficients a_ν *and the quantities*

$$b_k=\frac{(-1)^k}{m!}p_m^{(k)}(1), \qquad k=0,1,\dots,m, \tag{2.10}$$

(which uniquely determine p_m *in* (2.8)) *are obtained uniquely from the linear system*

$$\mathscr{L}_0(t^{m+1}p)=\mathscr{L}(t^{m+1}p) \qquad \text{all } p\in\mathbb{P}_{n+m}, \tag{2.11}$$

where

$$\mathscr{L}_0(g)=\sum_{k=0}^{m}b_k g^{(m-k)}(1)+\sum_{\nu=1}^{n}a_\nu g(\tau_\nu), \qquad \tau_\nu=\tau_\nu^{(n)}. \tag{2.12}$$

Proof. Substituting (2.1) in (2.2), and observing that $0<\tau_\nu<1$, gives

$$\begin{gathered}\int_0^1 t^j p_m(t)\,dt+\sum_{\nu=1}^{n}a_\nu\int_0^{\tau_\nu}t^j(\tau_\nu-t)^m\,dt=\int_0^1 t^j f(t)\,dt,\\ j=0,1,\dots,2n+m.\end{gathered} \tag{2.13}$$

Changing variables, $t=\tau\tau_\nu$, in the ν-th integral of the summation, one obtains

$$\begin{aligned}\int_0^{\tau_\nu}t^j(\tau_\nu-t)^m\,dt&=\tau_\nu^{m+j+1}\int_0^1\tau^j(1-\tau)^m\,d\tau\\ &=\frac{j!\,m!}{(m+j+1)!}\tau_\nu^{m+j+1}.\end{aligned} \tag{2.14}$$

Using m integrations by parts in the first integral of (2.13) yields

$$\int_0^1 t^j p_m(t)\,dt = \frac{j!\,m!}{(m+j+1)!}\sum_{k=0}^{m} b_k \left[\frac{d^{m-k}}{dt^{m-k}} t^{m+1+j}\right]_{t=1}, \tag{2.15}$$

where b_k is defined in (2.10). Inserting (2.14) and (2.15) in (2.13) and dividing through by $j!\,m!/(m+j+1)!$ gives

$$\mathscr{L}_0(t^{m+1}\cdot t^j)=\mu_j, \qquad j=0,1,\ldots,2n+m,$$

where μ_j is defined by (2.4) and $\mathscr{L}_0$ by (2.12). Therefore, using (2.5) and the linearity of $\mathscr{L}_0$ and $\mathscr{L}$,

$$\mathscr{L}_0(t^{m+1}p)=\mathscr{L}(t^{m+1}p), \qquad \text{all } p\in\mathbb{P}_{2n+m}. \tag{2.16}$$

Thus, the moment equations (2.2) and Eqs. (2.16) are equivalent.

Let now π_n denote the "knot polynomial"

$$\pi_n(t)=\prod_{\nu=1}^{n}(t-\tau_\nu) \tag{2.17}$$

having the knots τ_ν of the spline (2.8) as zeros. Then, by the definition of the inner product (2.6) we have, for any $q\in\mathbb{P}_{n-1}$,

$$(\pi_n, q)=\mathscr{L}(t^{m+1}(1-t)^{m+1}\pi_n\cdot q)=\mathscr{L}_0(t^{m+1}(1-t)^{m+1}\pi_n\cdot q), \tag{2.18}$$

by (2.16), since $(1-t)^{m+1}\pi_n\cdot q\in\mathbb{P}_{2n+m}$. Therefore, $(\pi_n, q)=0$ by the definition (2.12) of $\mathscr{L}_0$ and the fact that $\pi_n(\tau_\nu)=0$, $\nu=1,2,\ldots,n$. It follows that the knots τ_ν must be the zeros of the orthogonal polynomial $\pi_n(\cdot;\mathscr{L})$ of (2.7). This proves the necessity of the condition asserted in Theorem 2.1. Furthermore, the system (2.11) is a trivial consequence of (2.16); with $\tau_\nu=\tau_\nu^{(n)}$ determined, (2.11) is essentially a confluent Vandermonde system, hence nonsingular.

To prove the sufficiency of the condition, together with (2.11), we must show that they imply (2.16). Thus, let $p\in\mathbb{P}_{2n+m}$ be an arbitrary polynomial of degree $\leqq 2n+m$. Let q and r be the quotient and remainder of p upon division by $(1-t)^{m+1}\pi_n(t)$, where $\pi_n(\cdot)=\pi_n(\cdot;\mathscr{L})$,

$$p(t)=(1-t)^{m+1}\pi_n(t)\,q(t)+r(t), \qquad q\in\mathbb{P}_{n-1},\ r\in\mathbb{P}_{n+m}. \tag{2.19}$$

Then,

$$\begin{aligned}
\mathscr{L}(t^{m+1}p)&=\mathscr{L}(t^{m+1}(1-t)^{m+1}\pi_n\cdot q)+\mathscr{L}(t^{m+1}r)\\
&=\mathscr{L}(t^{m+1}r) \qquad [\text{by } (2.7)]\\
&=\mathscr{L}_0(t^{m+1}r) \qquad [\text{by } (2.11)]\\
&=\mathscr{L}_0(t^{m+1}p)-\mathscr{L}_0(t^{m+1}(1-t)^{m+1}\pi_n\cdot q) \qquad [\text{by } (2.19)]\\
&=\mathscr{L}_0(t^{m+1}p) \qquad [\text{since } \pi_n(\tau_\nu)=0].
\end{aligned}$$

This proves (2.16). □

The solution of Problem I* can be effected similarly, if one observes, in view of $0<\tau_\nu<1$, that

$$s_{n,m}^{(k)}(1)=p_m^{(k)}(1), \qquad k=0,1,\ldots,m. \tag{2.20}$$

By (2.3), therefore, $p_m^{(k)}(1)=f^{(k)}(1)$, $k=0,1,\ldots,m$, so that the moment equations in question can now be written as

$$\sum_{\nu=1}^{n} a_\nu \int_0^{\tau_\nu} t^j(\tau_\nu-t)^m\,dt=\int_0^1 t^j\left[f(t)-\sum_{k=0}^{m}\frac{f^{(k)}(1)}{k!}(t-1)^k\right]dt,\qquad j=0,1,\ldots,2n-1. \tag{2.21}$$

In analogy to (2.4) we define

$$\mu_j^*=\frac{(m+j+1)!}{m!\,j!}\int_0^1 t^j\left[f(t)-\sum_{k=0}^{m}\frac{f^{(k)}(1)}{k!}(t-1)^k\right]dt,\qquad j=0,1,\ldots,2n-1, \tag{2.22}$$

which gives rise to the linear functional $\mathscr{L}^*$ on polynomials of the form $t^{m+1}p(t)$, $p\in\mathbb{P}_{2n-1}$, defined by

$$\mathscr{L}^*(t^{m+1}\cdot t^j)=\mu_j^*,\qquad j=0,1,\ldots,2n-1, \tag{2.23}$$

and the inner product

$$(p,q)^*=\mathscr{L}^*(t^{m+1}p\cdot q),\qquad p\cdot q\in\mathbb{P}_{2n-1}. \tag{2.24}$$

The orthogonal polynomial $\pi_n^*(\cdot)=\pi_n(\cdot;\mathscr{L}^*)$ is now defined by

$$\deg\pi_n^*=n,\qquad \pi_n^*(t)=t^n+\ldots,$$
$$(\pi_n^*,q)^*=0,\qquad \text{all } q\in\mathbb{P}_{n-1}. \tag{2.25}$$

Then the result for Problem I*, analogous to Theorem 2.1, is given by the following

Theorem 2.2. *There exists a unique spline function on* $[0,1]$,

$$s_{n,m}^*(t)=p_m^*(t)+\sum_{\nu=1}^{n} a_\nu^*(\tau_\nu^*-t)_+^m,\qquad 0<\tau_\nu^*<1,\ \tau_\nu^*\neq\tau_\mu^*\ \textit{for}\ \nu\neq\mu, \tag{2.26}$$

satisfying (2.3) *and the* $2n$ *moment equations of Problem I* if and only if the orthogonal polynomial* $\pi_n^*(\cdot)=\pi_n(\cdot;\mathscr{L}^*)$ *in* (2.25) *exists uniquely and has* n *distinct real zeros* $\tau_\nu^{(n)*}$, $\nu=1,2,\ldots,n$, *all contained in the open interval* $(0,1)$. *The knots* τ_ν^* *in* (2.26) *are then precisely these zeros,*

$$\tau_\nu^*=\tau_\nu^{(n)*},\qquad \nu=1,2,\ldots,n, \tag{2.27}$$

the polynomial p_m^* *is given by*

$$p_m^*(t)=\sum_{k=0}^{m}\frac{f^{(k)}(1)}{k!}(t-1)^k, \tag{2.28}$$

and the coefficients a_ν^* *are obtained uniquely from the linear system*

$$\mathscr{L}_0^*(t^{m+1}p)=\mathscr{L}^*(t^{m+1}p),\qquad \textit{all}\ p\in\mathbb{P}_{n-1}, \tag{2.29}$$

where

$$\mathscr{L}_0^*(g)=\sum_{\nu=1}^{n} a_\nu^* g(\tau_\nu^*), \qquad \tau_\nu^*=\tau_\nu^{(n)*}. \tag{2.30}$$

The proof is entirely analogous to the proof of Theorem 2.1 and is omitted.

The functions $s_{n,m}$ and $s_{n,m}^*$ of Theorems 2.1 and 2.2 may be thought of as solutions of finite moment problems in terms of spline functions.

2.2. *Solution of Problems I and I* by Gauss-Christoffel Quadrature*

While the solution of Problems I, I* given in the previous subsection has some intrinsic mathematical interest, it is suspect, computationally, because of its reliance on the "moments" (2.4) and (2.22), which are likely to create ill-conditioning. For constructive purposes, it is better to reduce these problems to Gauss-Christoffel quadrature with respect to an absolutely continuous measure, as was similarly done in [4, 6]. This requires more regularity of f; we shall assume, in fact, that $f\in C^{m+1}[0,1]$. (This hypothesis could be slightly weakened.) We also assume that $f^{(k)}(1)$, $k=0,1,\dots,m$, are known, and that $f\notin \mathbb{P}_m$ (otherwise, trivially, $s_{n,m}\equiv f$).

Again, we first consider Problem I. Applying (2.14), (2.15) and $m+1$ integrations by parts to the last integral in the moment equations (2.13) now results in

$$\begin{aligned}\sum_{k=0}^{m} b_k\left[\frac{d^{m-k}}{dt^{m-k}}t^{m+1+j}\right]_{t=1}&+\sum_{\nu=1}^{n} a_\nu \tau_\nu^{m+1+j}\\ &=\sum_{k=0}^{m}\phi_k\left[\frac{d^{m-k}}{dt^{m-k}}t^{m+1+j}\right]_{t=1}+\frac{(-1)^{m+1}}{m!}\int_0^1 f^{(m+1)}(t)\,t^{m+1+j}\,dt,\\ &\qquad j=0,1,\dots,2n+m,\end{aligned} \tag{2.31}$$

where

$$b_k=\frac{(-1)^k p_m^{(k)}(1)}{m!}, \qquad \phi_k=\frac{(-1)^k f^{(k)}(1)}{m!}, \qquad k=0,1,\dots,m. \tag{2.32}$$

Defining the measure

$$d\lambda_m(t)=\frac{(-1)^{m+1}}{m!}f^{(m+1)}(t)\,dt \qquad \text{on } [0,1], \tag{2.33}$$

we can rewrite (2.31), similarly as in (2.16), in the form

$$\mathscr{L}_0(t^{m+1}p)=\mathscr{L}(t^{m+1}p), \qquad \text{all } p\in\mathbb{P}_{2n+m}, \tag{2.34}$$

where $\mathscr{L}_0$ is defined in (2.12), but $\mathscr{L}$ is now defined by

$$\mathscr{L}(g)=\sum_{k=0}^{m}\phi_k g^{(m-k)}(1)+\int_0^1 g(t)\,d\lambda_m(t). \tag{2.35}$$

The resolution of (2.34) is now verbatim the same as in the proof of Theorem 2.1, the inner product again being defined as in (2.6), but now with $\mathscr{L}$ given in (2.35). This yields

Theorem 2.3. *Assume that $f \in C^{m+1}[0,1]$. There exists a unique spline function* (2.8) *on* $[0,1]$ *satisfying the* $2n+m+1$ *moment equations* (2.2) *of Problem I if and only if the orthogonal polynomial* $\pi_n(\cdot)=\pi_n(\cdot;\mathscr{L})$ *in* (2.7) *relative to the inner product* (2.6), (2.35) *exists uniquely and has n distinct real zeros* $\tau_\nu^{(n)}$, $\nu=1,2,\ldots,n$, *all contained in the open interval* $(0,1)$. *The knots* τ_ν *in* (2.8) *are then precisely these zeros,*

$$\tau_\nu=\tau_\nu^{(n)}, \qquad \nu=1,2,\ldots,n, \tag{2.36}$$

while the coefficients a_ν, *and the quantities* b_k *in* (2.32) (*which uniquely determine* p_m *in* (2.8)), *are obtained uniquely from the linear system*

$$\mathscr{L}_0(t^{m+1}p)=\mathscr{L}(t^{m+1}p), \qquad \text{all } p\in \mathbb{P}_{n+m}, \tag{2.37}$$

where $\mathscr{L}_0$, $\mathscr{L}$ *are defined, respectively, by* (2.12) *and* (2.35).

The result of Theorem 2.3 has been announced without proof in [5, § 3.3]. It can also be interpreted in terms of the generalized Gauss-Lobatto quadrature formula (relative to the measure $d\lambda_m$ in (2.33)),

$$\begin{aligned}\int_0^1 g(t)\,d\lambda_m(t)=&\sum_{k=0}^{m}[A_k g^{(k)}(0)+B_k g^{(k)}(1)]\\&+\sum_{\nu=1}^{n}\lambda_\nu^{(n)} g(\tau_\nu^{(n)})+R_{n,m}(g;d\lambda_m),\end{aligned} \tag{2.38}$$

where

$$R_{n,m}(g;d\lambda_m)=0, \qquad \text{all } g\in\mathbb{P}_{2n+2m+1}. \tag{2.39}$$

This quadrature formula, in turn, is known to be related to the Gauss-Christoffel quadrature formula

$$\int_0^1 g(t)\,d\sigma_m(t)=\sum_{\nu=1}^{n}\sigma_\nu^{(n)} g(\tau_\nu^{(n)})+R_n(g;d\sigma_m), \qquad R_n(\mathbb{P}_{2n-1};d\sigma_m)=0, \tag{2.40}$$

with respect to the measure

$$d\sigma_m(t)=t^{m+1}(1-t)^{m+1}\,d\lambda_m(t) \qquad \text{on } [0,1]. \tag{2.41}$$

Indeed, the nodes $\tau_\nu^{(n)}$ in (2.38) and (2.40) are the same (equal to the zeros of $\pi_n(\cdot;d\sigma_m)$), while the weights $\lambda_\nu^{(n)}$ in (2.38) are expressible in terms of those in (2.40) by

$$\lambda_\nu^{(n)}=[\tau_\nu^{(n)}(1-\tau_\nu^{(n)})]^{-(m+1)}\sigma_\nu^{(n)}, \qquad \nu=1,2,\ldots,n. \tag{2.42}$$

Furthermore, the coefficients A_k, B_k in (2.38) can be obtained from the linear system

$$R_{n,m}(p;d\lambda_m)=0, \qquad \text{all } p\in\mathbb{P}_{2m+1}. \tag{2.43}$$

Now we note that the inner product (2.6), in view of (2.35), can be written in the form

$$(p,q)=\int_0^1 t^{m+1}(1-t)^{m+1}p(t)q(t)\,d\lambda_m(t)=\int_0^1 p(t)q(t)\,d\sigma_m(t). \tag{2.6'}$$

Therefore, the knots τ_ν in (2.36) are precisely the nodes in (2.40), hence those in (2.38). Putting $g(t)=t^{m+1}p(t)$, $p\in\mathbb{P}_{2n+m}$, in (2.38) and noting (2.39) yields

$$\sum_{k=0}^{m} B_k \frac{d^k}{dt^k}[t^{m+1}p(t)]_{t=1} + \sum_{\nu=1}^{n} \lambda_\nu^{(n)}[\tau_\nu^{(n)}]^{m+1} p(\tau_\nu^{(n)}) = \int_0^1 t^{m+1}p(t)\,d\lambda_m(t), \qquad \text{all } p\in\mathbb{P}_{2n+m},$$

which is identical to (2.34), if we identify

$$b_k-\phi_k=B_{m-k}, \qquad k=0,1,\dots,m; \qquad a_\nu=\lambda_\nu^{(n)}, \qquad \nu=1,2,\dots,n.$$

Since under the assumptions of Theorem 2.3 the solution of (2.34) is unique, we have shown the following

Corollary 1 to Theorem 2.3. *If the conditions of Theorem 2.3 are satisfied, then the spline function* (2.8) *solving Problem I is given by*

$$\tau_\nu=\tau_\nu^{(n)}, \qquad a_\nu=\lambda_\nu^{(n)}, \qquad \nu=1,2,\dots,n, \tag{2.44}$$

where $\tau_\nu^{(n)}$ are the interior nodes of the generalized Gauss-Lobatto quadrature formula (2.38) [*or the nodes of the Gauss-Christoffel formula* (2.40)] *and $\lambda_\nu^{(n)}$ the corresponding weights in* (2.38) [*or* (2.42)], *while*

$$p_m^{(k)}(1)=f^{(k)}(1)+(-1)^k m!\,B_{m-k}, \qquad k=0,1,\dots,m, \tag{2.45}$$

where B_{m-k} is the coefficient of $g^{(m-k)}(1)$ in the Gauss-Lobatto formula (2.38).

We remark that the conditions of Theorem 2.3 are satisfied for each $m=0,1,2,\dots$ if f is completely monotonic on [0, 1] (cf. [8, p. 145ff.]), since $d\lambda_m$, and hence also $d\sigma_m$, is then a positive measure. We have, moreover, the following

Corollary 2 to Theorem 2.3. *If f is completely monotonic on* [0, 1] *and for some $m\geqq 0$,*

$$m!\,B_{m-\mu}+(-1)^\mu f^{(\mu)}(1)>0, \qquad \mu=0,1,\dots,m, \tag{2.46}$$

then so is $s_{n,m}$ for each $n\geqq 1$; more precisely,

$$(-1)^k s_{n,m}^{(k)}(t)\begin{cases} >0 & \text{if } k=0,1,\dots,m,\\ =0 & \text{if } k>m,\end{cases} \tag{2.47}$$

for each $t\in[0,1]$ for which $s_{n,m}^{(k)}(t)$ is defined.

Proof. The assumption (2.46) implies $(-1)^\mu p_m^{(\mu)}(1)>0$, $\mu=0,1,\dots,m$, hence the positivity on [0, 1] of $(-1)^k p_m^{(k)}(t)=(-1)^k\left[\sum_{\mu=0}^{m}(-1)^\mu \mu!^{-1} p_m^{(\mu)}(1)(1-t)^\mu\right]^{(k)}$ for $k=0,1,\dots,m$. Since $a_\nu>0$, by (2.44), and $(-1)^k[(\tau_\nu-t)_+^m]^{(k)}\geqq 0$, $k=0,1,\dots,m$, whenever the derivative exists, the assertion (2.47) follows. □

We note that (2.46) restricts only those $B_0, B_1, \ldots, B_m$ that are negative. In the case of the infinite interval $[0, \infty]$, considered in [6], the property (2.47) (with $\geqq$ in place of $>$) follows directly from (2.8), since $p_m(t) \equiv 0$.

Turning now to Problem I*, we note that (2.20) again implies $p_m^{(k)}(1) = f^{(k)}(1)$, hence $b_k = \phi_k$, $k = 0, 1, \ldots, m$. The moment equations in question thus simplify to

$$\mathscr{L}_0^*(t^{m+1} p) = \mathscr{L}^*(t^{m+1} p), \quad \text{all } p \in \mathbb{P}_{2n-1}, \tag{2.48}$$

where $\mathscr{L}_0^*$ is given by (2.30) and $\mathscr{L}^*$ by

$$\mathscr{L}^*(g) = \int_0^1 g(t)\, d\lambda_m(t). \tag{2.49}$$

The analogue of Theorem 2.2, therefore, is as follows.

Theorem 2.4. *Assume that* $f \in C^{m+1}[0, 1]$. *There exists a unique spline function* (2.26) *on* $[0, 1]$ *satisfying* (2.3) *and the* $2n$ *moment equations of Problem I* if and only if the orthogonal polynomial* $\pi_n^*(\cdot) = \pi_n(\cdot\,; \mathscr{L}^*)$ *in* (2.25) *relative to the inner product* (2.24), (2.49) *exists uniquely and has* n *distinct real zeros* $\tau_\nu^{(n)*}$, $\nu = 1, 2, \ldots, n$, *all contained in the open interval* $(0, 1)$. *The knots* τ_ν^* *in* (2.26) *are then precisely these zeros,*

$$\tau_\nu^* = \tau_\nu^{(n)*}, \quad \nu = 1, 2, \ldots, n, \tag{2.50}$$

the polynomial p_m^* *is given by*

$$p_m^*(t) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!} (t-1)^k, \tag{2.51}$$

and the coefficients a_ν^* *are obtained uniquely from the linear system*

$$\mathscr{L}_0^*(t^{m+1} p) = \mathscr{L}^*(t^{m+1} p), \quad \text{all } p \in \mathbb{P}_{n-1}, \tag{2.52}$$

where $\mathscr{L}_0^*$, $\mathscr{L}^*$ *are defined, respectively, by* (2.30) *and* (2.49).

Underlying Theorem 2.4 is now the generalized Gauss-Radau quadrature formula,

$$\begin{gathered}\int_0^1 g(t)\, d\lambda_m(t) = \sum_{k=0}^{m} A_k^* g^{(k)}(0) + \sum_{\nu=1}^{n} \lambda_\nu^{(n)*} g(\tau_\nu^{(n)*}) + R_{n,m}^*(g; d\lambda_m), \\ R_{n,m}^*(g; d\lambda_m) = 0, \quad \text{all } g \in \mathbb{P}_{2n+m},\end{gathered} \tag{2.53}$$

or the related Gauss-Christoffel formula

$$\int_0^1 g(t)\, d\sigma_m^*(t) = \sum_{\nu=1}^{n} \sigma_\nu^{(n)*} g(\tau_\nu^{(n)*}) + R_n^*(g; d\sigma_m^*), \quad R_n^*(\mathbb{P}_{2n-1}; d\sigma_m^*) = 0 \tag{2.54}$$

for the measure

$$d\sigma_m^*(t) = t^{m+1} d\lambda_m(t) \quad \text{on } [0, 1]. \tag{2.55}$$

Again, the nodes $\tau_\nu^{(n)*}$ in (2.53) and (2.54) are identical, whereas

$$\lambda_\nu^{(n)*} = [\tau_\nu^{(n)*}]^{-(m+1)} \sigma_\nu^{(n)*}, \quad \nu = 1, 2, \ldots, n. \tag{2.56}$$

One has, in fact,

Corollary 1 to Theorem 2.4. *If the conditions of Theorem 2.4 are satisfied, then the spline function* (2.26) *solving Problem I* is given by* p_m^* *as in* (2.51) *and by*

$$\tau_\nu^* = \tau_\nu^{(n)*}, \quad a_\nu^* = \lambda_\nu^{(n)*}, \quad \nu = 1, 2, \dots, n, \tag{2.57}$$

where $\tau_\nu^{(n)*}$ *are the interior nodes of the generalized Gauss-Radau formula* (2.53) [*or the nodes of the Gauss-Christoffel formula* (2.54)] *and* $\lambda_\nu^{(n)*}$ *the corresponding weights in* (2.53) [*or* (2.56)].

Corollary 2 to Theorem 2.4. *If f is completely monotonic on* [0, 1] *then so is* $s_{n,m}^*$ *for each* $n \geqq 1$, $m \geqq 0$; *more precisely,*

$$(-1)^k s_{n,m}^{*(k)}(t) \begin{cases} >0 & \text{if } k=0,1,\dots,m, \\ =0 & \text{if } k>m, \end{cases} \tag{2.58}$$

for each $t \in [0, 1]$ *for which* $s_{n,m}^{*(k)}(t)$ *is defined.*

The proofs are analogous to the proofs of Corollaries 1 and 2 to Theorem 2.3 and are omitted.

To obtain the Gauss-Christoffel formulae in question, one must be able to generate the orthogonal polynomials relative to the measures $d\sigma_m$ and $d\sigma_m^*$ in (2.41) and (2.55), respectively. For this, the methods discussed in [2] and [3] (see also [1, § 5]) are often helpful.

3. Error and Convergence of Approximation

Similarly as in [6], the error of the spline approximants $s_{n,m}$ and $s_{n,m}^*$ constructed in Sect. 2 can be expressed in terms of the quadrature error of the generalized Gauss-Lobatto and Gauss-Radau formulae (2.38) and (2.53), respectively, when applied to a special function. This is the content of the next two theorems.

Theorem 3.1. *Assume the conditions of Theorem 2.3 are satisfied. Then, for any x with* $0 < x < 1$, *the spline function* $s_{n,m}$ *in* (2.8), *solving Problem I, approximates f with an error given by*

$$f(x) - s_{n,m}(x) = R_{n,m}(\rho_x; d\lambda_m), \tag{3.1}$$

where $R_{n,m}(\cdot; d\lambda_m)$ *is the remainder term in the generalized Gauss-Lobatto quadrature formula* (2.38) (*relative to the measure* $d\lambda_m$ *in* (2.33)) *and* ρ_x *is given by*

$$\rho_x(t) = (t - x)_+^m, \quad 0 \leqq t \leqq 1. \tag{3.2}$$

Alternatively, we have

$$f(x) - s_{n,m}(x) = R_n(\sigma_x; d\sigma_m), \tag{3.3}$$

where $R_n(\cdot; d\sigma_m)$ *is the remainder term in the Gauss-Christoffel quadrature formula* (2.40) (*relative to the measure* $d\sigma_m$ *in* (2.41)) *and* σ_x *is given by*

$$\sigma_x(t) = t^{-(m+1)}(1-t)^{-(m+1)}[\rho_x(t) - q_{2m+1}(\rho_x; t)], \tag{3.4}$$

$q_{2m+1}(\rho_x;\cdot)$ being the polynomial of degree $\leq 2m+1$ interpolating to ρ_x and its first m derivatives $\rho_x^{(k)}$, $k=1,2,\ldots,m$, at $t=0$ and $t=1$.

Proof. By Taylor's theorem,

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(1)(x-1)^k + \frac{1}{m!}\int_1^x (x-t)^m f^{(m+1)}(t)\,dt \\ &= \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(1)(x-1)^k + \frac{(-1)^{m+1}}{m!}\int_x^1 (t-x)^m f^{(m+1)}(t)\,dt \\ &= \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(1)(x-1)^k + \int_0^1 \rho_x(t)\,d\lambda_m(t). \end{aligned}$$

By (2.44),

$$s_{n,m}(x) = \sum_{k=0}^{m} \frac{1}{k!} p_m^{(k)}(1)(x-1)^k + \sum_{\nu=1}^{n} \lambda_\nu^{(n)} (\tau_\nu^{(n)} - x)_+^m.$$

Subtracting this from the preceding equation gives

$$\begin{aligned} f(x) - s_{n,m}(x) &= \int_0^1 \rho_x(t)\,d\lambda_m(t) + \sum_{k=0}^{m} \frac{1}{k!}[f^{(k)}(1) - p_m^{(k)}(1)](x-1)^k \\ &\quad - \sum_{\nu=1}^{n} \lambda_\nu^{(n)} (\tau_\nu^{(n)} - x)_+^m, \end{aligned}$$

which, by virtue of (2.45) and (3.2), yields

$$f(x) - s_{n,m}(x) = \int_0^1 \rho_x(t)\,d\lambda_m(t) - \sum_{k=0}^{m} \frac{m!}{k!} B_{m-k}(1-x)^k - \sum_{\nu=1}^{n} \lambda_\nu^{(n)} \rho_x(\tau_\nu^{(n)}).$$

But

$$\rho_x^{(k)}(0) = 0, \qquad \rho_x^{(k)}(1) = \frac{m!}{(m-k)!}(1-x)^{m-k}, \qquad k = 0, 1, \ldots, m,$$

so that

$$\begin{aligned} f(x) - s_{n,m}(x) &= \int_0^1 \rho_x(t)\,d\lambda_m(t) - \sum_{k=0}^{m} B_{m-k}\,\rho_x^{(m-k)}(1) - \sum_{\nu=1}^{n} \lambda_\nu^{(n)} \rho_x(\tau_\nu^{(n)}) \\ &= R_{n,m}(\rho_x; d\lambda_m), \end{aligned}$$

as claimed in (3.1).

To prove (3.3), it suffices to observe that for any function h that has zeros of multiplicity $m+1$ at $t=0$ and $t=1$ one obtains from (2.38), (2.40) and (2.42), by putting $g(t) = t^{-(m+1)}(1-t)^{-(m+1)}h(t)$ in (2.40), that

$$R_n(t^{-(m+1)}(1-t)^{-(m+1)}h; d\sigma_m) = R_{n,m}(h; d\lambda_m). \tag{3.5}$$

In particular, for $h(t) = \rho_x(t) - q_{2m+1}(\rho_x; t)$, since $R_{n,m}(q_{2m+1}; d\lambda_m) = 0$, one gets $R_{n,m}(\rho_x; d\lambda_m) = R_{n,m}(\rho_x - q_{2m+1}; d\lambda_m) = R_n(\sigma_x; d\sigma_m)$, with σ_x given by (3.4). □

Theorem 3.2. *Assume the conditions of Theorem 2.4 are satisfied. Then, for any x with $0<x<1$, the spline function $s_{n,m}^*$ in (2.26), solving Problem I*, approximates f with an error given by*

$$f(x)-s^*_{n,m}(x)=R^*_{n,m}(\rho_x; d\lambda_m), \tag{3.6}$$

where $R^*_{n,m}(\cdot; d\lambda_m)$ *is the remainder term in the generalized Gauss-Radau quadrature formula* (2.53) (*relative to the measure* $d\lambda_m$ *in* (2.33)) *and* ρ_x *is given by* (3.2). *Alternatively, we have*

$$f(x)-s^*_{n,m}(x)=R^*_n(\sigma^*_x; d\sigma^*_m), \tag{3.7}$$

where $R^*_n(\cdot; d\sigma^*_m)$ *is the remainder term in the Gauss-Christoffel quadrature formula* (2.54) (*relative to the measure* $d\sigma^*_m$ *in* (2.55)) *and* σ^*_x *is given by*

$$\sigma^*_x(t)=t^{-(m+1)}\rho_x(t)=t^{-(m+1)}(t-x)^m_+, \qquad 0\leqq t\leqq 1. \tag{3.8}$$

Proof. Equation (3.6) is proved similarly as Eq. (3.1) in Theorem 3.1. The alternative formula (3.7) follows readily from $R^*_{n,m}(\rho_x; d\lambda_m)=R^*_n(\sigma^*_x; d\sigma^*_m)$. □

If $f\in C^{m+1}[0,1]$ is such that $d\lambda_m$ in (2.33) is a positive measure (for example, if f is completely monotonic on $[0,1]$), then the approximations $s_{n,m}$ and $s^*_{n,m}$ exist uniquely by Theorems 2.3 and 2.4, respectively. Moreover, for fixed $m>0$ and x, with $0<x<1$, we have

$$R_n(\sigma_x; d\sigma_m)\to 0 \qquad \text{as } n\to\infty$$

since σ_x is continuous on $[0,1]$ and $d\sigma_m$ is also a positive measure. Therefore, by (3.3), we have pointwise convergence $s_{n,m}\to f$ as $n\to\infty$. The analogous fact for $s^*_{n,m}$ follows likewise from (3.7) and the continuity of σ^*_x on $[0,1]$. Thus, we have

Theorem 3.3. *If* $f\in C^{m+1}[0,1]$ *and* $d\lambda_m$ *in* (2.33) *is a positive measure, then the approximations* $s_{n,m}$ *and* $s^*_{n,m}$ *constructed in Sect. 2 converge pointwise to* f *in* $(0,1)$, *as* $n\to\infty$ *for fixed* $m>0$.

We finally note that the formulas (3.1) and (3.3), resp. (3.6) and (3.7), by differentiating them repeatedly with respect to x, yield representations for the errors $f^{(k)}-s^{(k)}_{n,m}$ and $f^{(k)}-s^{*(k)}_{n,m}$ in the derivatives, respectively.

4. Examples

We illustrate the spline approximations of Theorems 2.3 and 2.4 (or their corollaries) in the case of exponential and trigonometric functions. All computations reported on were carried out on the CDC 6500 computer in single precision (machine precision $\approx 3.55\times 10^{-15}$).

Example 4.1. $f(t)=e^{-ct}$, $0\leqq t\leqq 1$, $c>0$.

This is an example of a completely monotonic function, for which the associated measure (2.33) is thus positive; indeed,

$$d\lambda_m(t)=\frac{c^{m+1}}{m!}e^{-ct}\,dt \qquad \text{on } [0,1]. \tag{4.1}$$

Problems I, I* therefore have unique solutions by Theorems 2.3 and 2.4. In terms of the generalized Gauss-Lobatto formula

$$\int_0^1 g(t)e^{-ct}\,dt=\sum_{k=0}^{m}[\bar{A}_k g^{(k)}(0)+\bar{B}_k g^{(k)}(1)]+\sum_{\nu=1}^{n}\bar{\lambda}_\nu^{(n)} g(\tau_\nu^{(n)})+\bar{R}_{n,m}(g;e^{-ct}\,dt) \tag{4.2}$$

and the generalized Gauss-Radau formula

$$\int_0^1 g(t)e^{-ct}\,dt=\sum_{k=0}^{m}\bar{A}_k^* g^{(k)}(0)+\sum_{\nu=1}^{n}\bar{\lambda}_\nu^{(n)*} g(\tau_\nu^{(n)*})+\bar{R}_{n,m}^*(g;e^{-ct}\,dt), \tag{4.3}$$

we have from Corollary 1 to Theorem 2.3,

$$\begin{gathered}\tau_\nu=\tau_\nu^{(n)},\qquad a_\nu=\frac{c^{m+1}}{m!}\bar{\lambda}_\nu^{(n)},\qquad \nu=1,2,\dots,n,\\ p_m(t)=\frac{c^{m+1}}{m!}\sum_{k=0}^{m}\frac{m!}{k!}[c^{k-m-1}e^{-c}+\bar{B}_{m-k}](1-t)^k\end{gathered} \tag{4.4}$$

for the spline $s_{n,m}$ in (2.8), solving Problem I, and from Corollary 1 to Theorem 2.4,

$$\begin{gathered}\tau_\nu^*=\tau_\nu^{(n)*},\qquad a_\nu^*=\frac{c^{m+1}}{m!}\bar{\lambda}_\nu^{(n)*},\qquad \nu=1,2,\dots,n,\\ p_m^*(t)=\frac{c^{m+1}}{m!}\sum_{k=0}^{m}\frac{m!}{k!}c^{k-m-1}e^{-c}(1-t)^k\end{gathered} \tag{4.5}$$

for the spline $s_{n,m}^*$ in (2.26), solving Problem I*.

The Gaussian nodes and weights in (4.2) and (4.3) were obtained in the usual way (see, e.g., [2, p. 290]) in terms of the eigensystems of the Jacobi matrices $J_n(t^{m+1}(1-t)^{m+1}e^{-ct}dt)$ and $J_n(t^{m+1}e^{-ct}dt)$, respectively. The latter were generated from the Jacobi matrix $J_{n+2m+2}(e^{-ct}dt)$, resp. $J_{n+m+1}(e^{-ct}dt)$, by repeated application of the algorithms in [3, §4.1] corresponding to multiplication of a measure by $t(1-t)$ and t, respectively. (Alternatively, algorithms based on the QR algorithm, as in [7], could be used for the same purpose.) Finally, $J_{n+2m+2}(e^{-ct}dt)$ was computed by the discretized Stieltjes algorithm (see [2, §2.2]), the Fejér quadrature rule having been used as the modus of discretization.

As to the coefficients $\bar{A}_k$, $\bar{B}_k$ in the boundary terms of (4.2), they were computed from the linear system of equations

$$\bar{R}_{n,m}(p;e^{-ct}\,dt)=0,\qquad \text{all } p\in\mathbb{P}_{2m+1}, \tag{4.6}$$

where the first $2m+2$ orthogonal polynomials $\{\pi_k(\cdot;e^{-ct}dt)\}_{k\geq 0}$ (whose Jacobi matrix J_{n+2m+2} has already been generated!) were used as basis in the polynomial space $\mathbb{P}_{2m+1}$ of (4.6). The coefficients $\bar{A}_k^*$ in (4.3) are not needed.

The accuracy of the spline approximations $s_{n,m}$ and $s_{n,m}^*$ thus obtained is shown in Table 4.1 for $n=5$, 10, 20, 40; $m=0(1)3$; and $c=1, 2, 4$. Displayed are (two-digit approximations to) the respective maximum absolute errors on $[0, 1]$.

Table 4.1. Accuracy of the spline approximations $s_{n,m}$ and $s^*_{n,m}$ for Example 4.1. (Numbers in parentheses denote decimal exponents).

c	n	$\max_{0\le t\le 1}\lvert s_{n,m}(t)-e^{-ct}\rvert$				$\max_{0\le t\le 1}\lvert s^*_{n,m}(t)-e^{-ct}\rvert$			
		$m=0$	$m=1$	$m=2$	$m=3$	$m=0$	$m=1$	$m=2$	$m=3$
1	5	8.0 (−2)	2.4 (−3)	4.0 (−5)	9.7 (−7)	8.8 (−2)	3.3 (−3)	6.8 (−5)	2.4 (−6)
	10	4.6 (−2)	8.6 (−4)	8.6 (−6)	1.4 (−7)	4.8 (−2)	1.0 (−3)	1.2 (−5)	2.5 (−7)
	20	2.5 (−2)	2.6 (−4)	1.5 (−6)	1.5 (−8)	2.5 (−2)	2.9 (−4)	1.9 (−6)	2.1 (−8)
	40	1.3 (−2)	7.3 (−5)	2.4 (−7)	1.4 (−9)	1.3 (−2)	7.7 (−5)	2.7 (−7)	1.6 (−9)
2	5	1.3 (−1)	7.0 (−3)	2.1 (−4)	9.8 (−6)	1.3 (−1)	9.1 (−3)	3.8 (−4)	2.4 (−5)
	10	7.1 (−2)	2.4 (−3)	4.6 (−5)	1.5 (−6)	7.5 (−2)	2.8 (−3)	6.5 (−5)	2.6 (−6)
	20	3.9 (−2)	7.4 (−4)	8.4 (−6)	1.6 (−7)	4.0 (−2)	8.1 (−4)	1.0 (−5)	2.3 (−7)
	40	2.0 (−2)	2.1 (−4)	1.3 (−6)	1.4 (−8)	2.0 (−2)	2.2 (−4)	1.4 (−6)	1.7 (−8)
4	5	1.7 (−1)	1.6 (−2)	8.7 (−4)	7.8 (−5)	1.7 (−1)	1.9 (−2)	1.5 (−3)	2.5 (−4)
	10	1.0 (−1)	5.7 (−3)	2.0 (−4)	1.1 (−5)	1.1 (−1)	6.7 (−3)	2.7 (−4)	2.0 (−5)
	20	5.7 (−2)	1.8 (−3)	3.6 (−5)	1.3 (−6)	5.8 (−2)	2.0 (−3)	4.3 (−5)	1.8 (−6)
	40	3.0 (−2)	5.1 (−4)	5.7 (−6)	1.2 (−7)	3.0 (−2)	5.3 (−4)	6.2 (−6)	1.4 (−7)

For $m=0$, 1, and 3, the maxima are almost always attained at a knot of the spline, about half-way (or somewhat less) through the interval. The only exception observed was for $s^*_{n,m}$, $n=5$, $m=3$, $c=4$, where the maximum occurs at $t=0$. When $m=2$, the maxima are usually attained between two such knots. The linear system (4.6) (in the orthogonal basis mentioned) was found to be relatively well-conditioned, the worst condition number (occurring for $m=3$) being approx. 2.5×10^3.

It is seen that the approximation error is more easily reduced by increasing m rather than n. Also, the spline $s_{n,m}$ is only marginally more accurate than the spline $s^*_{n,m}$. The additional effort required in computing $s_{n,m}$, therefore, seems hardly justified, if uniform approximation is indeed the main objective. If moment-matching is more important, however, the spline $s_{n,m}$ would be preferable, as it matches $m+1$ additional moments.

The coefficients of p_m, i.e., the expressions in the brackets of (4.4), turned out to be positive for all values of m, n and c tried, so that the computed splines $s_{n,m}$ are completely monotonic in the sense of (2.47). The analogous property for $s^*_{n,m}$ follows from Corollary 2 to Theorem 2.4.

Example 4.2. $f(t)=\sin\frac{\pi}{2}t,\ 0\leqq t\leqq 1.$

Here, the function f, though not completely monotonic, still has derivatives that are all of constant sign on [0, 1]. Therefore, the measure $d\lambda_m$ in (2.33), i.e.,

$$d\lambda_m(t)=\frac{(-1)^{[m/2]+1}}{m!}\left(\frac{\pi}{2}\right)^{m+1}\begin{Bmatrix}\cos\frac{\pi}{2}t\\ \sin\frac{\pi}{2}t\end{Bmatrix}dt \quad \text{on } [0,1], \tag{4.7}$$

where the cosine or sine is taken according as m is even or odd, admits a unique system of (monic) orthogonal polynomials, and Problems I and I* both have unique solutions for each m and n. Observing that the substitution $t \to 1-t$ carries the cosine into the sine, and vice versa, it suffices to generate the orthogonal polynomials for one of the trigonometric measures only, say $\cos((\pi/2)t)dt$. If α_k^c, β_k^c, are the coefficients in the corresponding recurrence relation

$$\begin{aligned}\pi_{k+1}(t)&=(t-\alpha_k)\,\pi_k(t)-\beta_k\,\pi_{k-1}(t), \qquad k=0,1,2,\ldots,\\ \pi_{-1}(t)&=0, \qquad \pi_0(t)=1,\end{aligned} \tag{4.8}$$

then the coefficients α_k^s, β_k^s for the sine-measure are

$$\alpha_k^s=1-\alpha_k^c, \qquad \beta_k^s=\beta_k^c, \qquad k=0,1,2,\ldots. \tag{4.9}$$

A similar remark applies to the generalized Lobatto measures (2.41) [but not to the generalized Radau measures (2.55)]. The constants multiplying the trigonometric measures in (4.7), of course, simply give rise to analogous multiplicative constants in the quadrature rules (2.38) and (2.53).

Techniques similar to those in Example 4.1 were used to compute the desired spline approximants in the present example.

Table 4.2. Accuracy of the spline approximations $s_{n,m}$ and $s^*_{n,m}$ for Example 4.2.

n	$\max_{0\le t\le 1}\left\lvert s_{n,m}(t)-\sin\frac{\pi}{2}t\right\rvert$				$\max_{0\le t\le 1}\left\lvert s^*_{n,m}(t)-\sin\frac{\pi}{2}t\right\rvert$			
	$m=0$	$m=1$	$m=2$	$m=3$	$m=0$	$m=1$	$m=2$	$m=3$
5	1.4 (−1)	6.5 (−3)	1.7 (−4)	6.2 (−6)	1.5 (−1)	8.8 (−3)	2.7 (−4)	1.5 (−5)
10	8.4 (−2)	2.4 (−3)	3.7 (−5)	9.4 (−7)	8.8 (−2)	2.8 (−3)	5.0 (−5)	1.6 (−6)
20	4.6 (−2)	7.6 (−4)	6.8 (−6)	1.1 (−7)	4.7 (−2)	8.2 (−4)	8.2 (−6)	1.4 (−7)
40	2.4 (−2)	2.1 (−4)	1.1 (−6)	9.6 (−9)	2.4 (−2)	2.2 (−4)	1.2 (−6)	1.1 (−8)

Their accuracy is shown in Table 4.2; the error behaves rather similarly as the error in Example 4.1 for $c=2$.

References

1. Gautschi, W.: A survey of Gauss-Christoffel quadrature formulae. In: E.B. Christoffel, Butzer, P.L., Fehér, F. (eds.), pp. 72-147. Basel: Birkhäuser 1981
2. Gautschi, W.: On generating orthogonal polynomials. SIAM J. Sci. Stat. Comput. **3**, 289-317 (1982)
3. Gautschi, W.: An algorithmic implementation of the generalized Christoffel theorem. In: Numerische Integration. Hämmerlin, G. (ed.). Internat. Ser. Numer. Math., vol. 57, pp. 89-106. Basel: Birkhäuser 1982

4. Gautschi, W.: Discrete approximations to spherically symmetric distributions. Numer. Math. **44**, 53–60 (1984)
5. Gautschi, W.: Some new applications of orthogonal polynomials. In: Polynômes orthogonaux et applications. Brezinski, C., Draux, A., Magnus, A.P., Maroni, P., Ronveaux, A. (eds.). Lecture Notes Math., vol. 1171, pp. 63–73. Berlin-Heidelberg-New York-Tokyo: Springer 1985
6. Gautschi, W., Milovanović, G.V.: Spline approximations to spherically symmetric distributions. Numer. Math. **49**, 111–121 (1986)
7. Golub, G.H., Kautsky, J.: Calculation of Gauss quadratures with multiple free and fixed knots. Numer. Math. **41**, 147–163 (1983)
8. Widder, D.V.: The Laplace Transform. Princeton: University Press 1941

Received July 21, 1986/November 17, 1986